THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Tutorial 1

Date: 12 September, 2024

TA Information:
Stephen Liu
stephenliu@cuhk.edu.hk
LSB 232A

Announcement:

- HW1 on course website. due 24/9 23:59pm on Gradescope
- Quiz 1 will be on 19/9 during lecture period.

Field Axioms: $\forall a,b,c \in \mathbb{R}$

A1) $a+b \in \mathbb{R}$ whenever $a,b \in \mathbb{R}$

A2) $a+b = b+a$

A3) $(a+b)+c = a+(b+c)$

A4) $\exists\, 0 \in \mathbb{R}$ s.t. $a+0 = a = 0+a$

A5) for each $a \in \mathbb{R}$, $\exists -a \in \mathbb{R}$ s.t. $a+(-a) = 0 = (-a)+a$

M1) $a \cdot b \in \mathbb{R}$ whenever $a,b \in \mathbb{R}$

M2) $a \cdot b = b \cdot a$

M3) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M4) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \neq 0$ and $1 \cdot a = a = a \cdot 1$

M5) $\forall a \in \mathbb{R}$ with $a \neq 0$, $\exists \frac{1}{a} \in \mathbb{R}$ s.t. $a \cdot \left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right) \cdot a$

D) $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot c + c \cdot a$.

1. (Exercise 2.1.9 of [BS11]) Let $K = \{s + t\sqrt{2} : s, t \in \mathbb{Q}\}$. Show that K is closed under addition, multiplication, and contains multiplicative inverses. This shows that there is an ordered subfield $\mathbb{Q} \subset K \subset \mathbb{R}$ with the order of K inherited from $\mathbb{R}$.

Rmk: Use the field axioms and label when you are using them.

$\mathbb{Q} \subseteq \mathbb{R}$

Pf: Multiplication: let $x_1, x_2 \in K$. Then WTS $x_1 \cdot x_2 \in K$.

By def'n of K, write $x_1 = s_1 + t_1\sqrt{2}$, $x_2 = s_2 + t_2\sqrt{2}$ for $s_1, t_1, s_2, t_2 \in \mathbb{Q}$.

$$x_1 \cdot x_2 = (s_1 + t_1\sqrt{2}) \cdot (s_2 + t_2\sqrt{2})$$

$$\overset{(A1, M1, D, M2)}{=} s_1 s_2 + s_1 t_2 \sqrt{2} + s_2 t_1 \sqrt{2} + 2 t_1 t_2$$

$$\overset{(A2)}{=} s_1 s_2 + 2 t_1 t_2 + s_1 t_2 \sqrt{2} + s_2 t_1 \sqrt{2}$$

$$\overset{(A3, D)}{=} \underbrace{(s_1 s_2 + 2 t_1 t_2)}_{\in \mathbb{Q}} + \underbrace{(s_1 t_2 + s_2 t_1)}_{\in \mathbb{Q}} \sqrt{2} \in K.$$

3. Let A, B be nonempty subsets of $\mathbb{R}$. Denote by $A+B$ the set $\{a+b : a \in A, b \in B\}$. Show that $\sup(A+B) = \sup A + \sup B$.

Pf: $\sup(A+B) \leq \sup A + \sup B$:

Let $a \in A$, $b \in B$. We know that $a \leq \sup A$, $b \leq \sup B$ by definition.

So adding these two inequalities, we get

$$a+b \leq \sup A + \sup B.$$

Since a, b were arbitrary, the element $a+b \in A+B$ was arbitrarily chosen, and so the number $\sup A + \sup B$ is an u.b. of $A+B$.

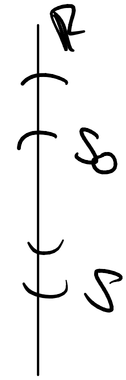
Then by def'n of sup. of $A+B$, we have

$$\sup(A+B) \leq \sup A + \sup B. \quad //.$$

4. (Exercise 2.1.11 of [BS11]). Let S be a bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S. Show that

$$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S.$$

"intuition"



Pf: $\inf S \leq \inf S_0$:

Since S is bounded, S_0 is also bounded.
and in addition, S_0 is non-empty.
So by Completeness Property $\inf S_0$, $\sup S_0$ exist in $\mathbb{R}$

Now suppose on the contrary that
$\inf S_0 < \inf S$.
Then this means that $\inf S$ is not a lower bound of S_0,
So we can find an $s_0 \in S_0$ s.t. (since $\inf S_0$ is greatest lower bound of S_0).
$\inf S_0 \leq s_0 < \inf S$.
But since $S_0 \subseteq S$, $s_0 \in S$, so we have a contradiction. //

6. Let $r \in \mathbb{R}$ be fixed. Determine the infimum and supremum of the set $X = \{|q - r| : q \in \mathbb{Q}\}$ if they exist.

Pf: First we'll show $\sup X$ does not exist by showing X does not have an upper bound:

let $0 \leq M \in \mathbb{R}$. WTS $\exists\, x \in X$ s.t. $M \leq x$. By A.P., there is an $n \in \mathbb{N}$ s.t. $M + r \leq n \iff M \leq n - r = |n-r|$. Since $n \in \mathbb{N} \subseteq \mathbb{Q}$, we have found such an element in X. $x = |n-r|$.

We'll show $\inf X = 0$:

- 0 is a lower bound: absolute value fn is non-neg.
- 0 is the greatest lower bound:

  use equivalent condition: $\forall \varepsilon > 0\ \exists\, x \in X$ s.t. $x < \varepsilon$.

let $\varepsilon > 0$ be given. By density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, q_\varepsilon \in \mathbb{Q}$ s.t. $|q_\varepsilon - r| < \varepsilon$. So setting $x := |q_\varepsilon - r|$, we are done. ✓

Note: $u = \inf S \iff \forall \varepsilon > 0\ \exists\, s \in S$ s.t. $s < u + \varepsilon$.